Chapter 6 The Prime Number Theorem 2019-20

[9 lectures]

The main reference for this material is *The Prime Number Theorem*, G. J. O. Jameson, LMS Student Texts, **53**, 2003.

The actual choice of function $F(s)$ and the manner of moving the line of integration in Step 5 is seen in *Introduction to Analytic and Probabilistic Number Theory*, G. Tenenbaum, Cambridge Studies in Advanced Mathematics, **46**, 1995.

The Prime Numbers and their Distribution, G. Tenenbaum & M. M. France, AMS Student Maths Library, vol 6. 1999 is a very brief overview of the material presented here.

Introduction. Plan of the proof of the Prime Number Theorem (PNT):

Aim The aim is to prove that $\psi(x) \sim x$ as $x \to \infty$, i.e. $\lim_{x\to\infty} \psi(x)/x = 1$ where $\psi(x) = \sum_{n\le x} \Lambda(n)$. It was a result in a previous Chapter that this is equivalent to $\pi(x) \sim x/\log x$ as $x \to \infty$, where $\pi(x) = \sum_{p\le x} 1$.

We will actually prove that $\psi(x) = x + O(x\mathcal{E}(x))$ with an explicitly given function $\mathcal{E}(x)$ satisfying $\mathcal{E}(x) \to 0$ as $x \to 0$. This is the Prime Number Theorem with an error term.

Step 1. Analytic Properties of the Riemann zeta function

[2 lectures]

The function $\zeta(s)$ has been defined by a series valid only for $\mathrm{Re}\, s > 1$. In the proof of the Prime Number Theorem we will need to define the zeta function for $s = 1 + it$, $t \neq 0$. So in this section we find a function holomorphic for $\mathrm{Re}\, s > 0$, except for a pole at $s = 1$, which agrees with the series definition of $\zeta(s)$ for $\mathrm{Re}\, s > 1$. That is, we find an ***analytic continuation*** of $\zeta(s)$ to $\mathrm{Re}\, s > 0$. See Background: Analytic Continuation notes for more discussion on this.

Step 2. Relate $\psi(x)$ to $\zeta'(s)/\zeta(s)$. **[1 lecture]**

We will show that

$$\int_1^x (\psi(t) - [t])\, dt = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^{s+1} ds}{s(s+1)} \tag{1}$$

where $c > 1$ and

$$F(s) = \frac{\zeta'(s)}{\zeta(s)} + \zeta(s)\,, \tag{2}$$

for $\operatorname{Re} s > 1$.

It is a question on a Problem Sheet to show that

$$\int_1^x [t]\, dt = \frac{1}{2}x^2 + O(x)\,,$$

in which case (1) becomes

$$\int_1^x \psi(t)\, dt = \frac{1}{2}x^2 - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, \frac{x^{s+1} ds}{s\,(s+1)} + O(x)\,. \tag{3}$$

Step 3. $\zeta(1+it) \neq 0$. **[2 lectures]**

The goal of $\psi(t) \sim t$ can be shown to imply $\int_1^x \psi(t)\, dt \sim x^2/2$. We already see $x^2/2$ on the right of (3) thus we expect the integral on the right hand side of (3) to be an *error term* in the final result, in particular to grow slower that $x^2/2$.

Yet this integral contains a factor x^{s+1}. In magnitude this is $|x^{s+1}| = x^{\operatorname{Re} s+1}$. If $\operatorname{Re} s + 1 > 2$, i.e. $\operatorname{Re} s > 1$, then $x^{\operatorname{Re} s+1}$ will dominate the $x^2/2$ term. So there is only a chance of this proof working if we can take $\operatorname{Re} s = 1$.

After Step 1 all terms $\zeta(s)$ and $\zeta'(s)$ are defined in $\operatorname{Re} s > 0$ but because of the zeta function in the denominator of $F(s)$ we need to know that $\zeta(s) \neq 0$ on $\operatorname{Re} s = 1$. We already know that this is the case for $\operatorname{Re} s > 1$.

Use is made here of the famous result, due to Mertens, 1898,

$$|\zeta(\sigma)|^3 \, |\zeta(\sigma+it)|^4 \, |\zeta(\sigma+2it)| \geq 1,$$

$\sigma > 1$, $t \neq 0$. This can be used to show that $\zeta(s) \neq 0$ for $\operatorname{Re} s = 1$.

This is, in fact, sufficient to prove the Prime Number Theorem in the form $\psi\,(t) \sim t$. But the proof of the Prime Number Theorem with an error term is conceptually easier if more detailed. For this we require results valid to the left of the vertical line $\operatorname{Re} s = 1$.

Step 4. Bounds on the Riemann zeta function. **[2 lectures]**

After Step 1 the functions $\zeta(s)$ and $\zeta'(s)$ are defined in $\operatorname{Re} s > 0$. We can relate these functions to partial sums of the infinite series that define $\zeta(s)$

and $\zeta'(s)$ in $\operatorname{Re} s > 1$. In this way we can give upper bounds on $\zeta(s)$ and $\zeta'(s)$ valid for $s = \sigma + it$ with $t > 2$ and $\sigma > 1 - 1/\log t$. That is, we have results valid to the left of the line $\operatorname{Re} s = 1$.

By Mertens' results, written as

$$|\zeta(\sigma+it)| \geq \frac{1}{|\zeta(\sigma)|^{3/4}\, |\zeta(\sigma+2it)|^{1/4}},$$

we can give lower bounds on $\zeta(s)$ valid for $s = \sigma + it$ with $t > 2$ and $\sigma > 1 + \delta(t)$ where $\delta(t)$ is essentially $c/\log^9 t$ for large t. Then, since we have a good bound on $\zeta'(s)$, it can be shown that $|\zeta(\sigma+it)|$ with $\sigma = 1 - \delta(t)$ differs little from the value with $\sigma = 1 + \delta(t)$. Thus we have lower bounds of $\zeta(s)$ valid for $s = \sigma + it$ with $t > 2$ and $\sigma > 1 - \delta(t)$. Again, we have results valid to the left of the line $\operatorname{Re} s = 1$.

Finally, the quality of the bounds found imply that

$$F(s) \leq \left| \frac{\zeta'(s)}{\zeta(s)} \right| + |\zeta(s)| \ll \log^9 t$$

for $s = \sigma + it$ with $t > 2$ and $\sigma \geq 1 - \delta(t)$.

Step 5 Moving the line of integration. **[1 lecture]**

With $T \geq 2$ to be chosen, truncate the integral in (1) at $\pm T$ and estimate the tail ends which are discarded by

$$\frac{1}{2\pi i} \int_{c+iT}^{c+i\infty} F(s)\, \frac{x^{s+1} ds}{s\,(s+1)} \ll x^{1+c} \int_T^{\infty} \frac{\log^9 t}{|c+it|\,|c+1+it|} dt \ll \frac{x^{1+c} \log^9 T}{T}.$$

This leaves us with the integral along the vertical straight line from $c - iT$ to $c + iT$. By an application of Cauchy's Theorem we show that

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s)\, \frac{x^{s+1} ds}{s\,(s+1)} = \frac{1}{2\pi i} \int_{1-\delta(T)+iT}^{1-\delta(T)-iT} F(s)\, \frac{x^{s+1} ds}{s\,(s+1)} + O\left(\frac{x^{1+c} \log^9 T}{T} \right), \tag{4}$$

where $\delta(t)$ is essentially $c/\log^9 t$ for large t. The final integral can be directly estimated and we conclude that

$$\int_1^x \psi(t)\, dt = \frac{1}{x} x^2 + O\left(x^2 \exp\left(-c \log^{1/10} x \right) \right), \tag{5}$$

for some constant $c > 0$.

Step 6 Final deduction. **[1/2 lecture]**

Finally we deduce the Prime Number Theorem, in the form

$$\psi(x) = x + O\left(x \exp\left(-c \log^{1/10} x\right)\right),$$

from (5). All that is required for this step is that $\psi(x) = \sum_{n \leq x} \Lambda(n)$ is an *increasing* function.